Block-Gauss-Seidel Immersed Boundary Method Accelerated by MUMPS

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MUMPS User Days.
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Outline

- Incompressible Navier-Stokes (NS) equations – a survey of existing solution methodologies
- Immersed boundary method (IBM) – general concepts and implementation
- From Vanka smoother to Block Gauss-Seidel method accelerated by MUMPS
- Incorporation of IBM into the Block Gauss-Seidel method
- Ongoing and future work

Incompressible Navier Stokes equations

Continuity: $\nabla \cdot \boldsymbol{u} = 0$

Momentum: $\dfrac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \mathrm{Re}^{-1} \nabla^2 \boldsymbol{u}$

- There is no separate equation for the pressure field
- Pressure is defined up to a constant
- Pressure field constitutes the Lagrange multiplier constraint to satisfy the continuity equation
- Generally speaking, no boundary condition is required for pressure

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Solution of Incompressible Navier-Stokes (NS) Equations

- Segregated approach (SIMPLE, Projection and their derivatives)

$$\frac{3\boldsymbol{u}^* - 4\boldsymbol{u}^n + \boldsymbol{u}^{n-1}}{2\Delta t} = -\nabla p^n + \frac{1}{Re}\nabla^2\boldsymbol{u}^* - (\boldsymbol{u}\cdot\nabla\boldsymbol{u})^n,$$

$$\nabla^2 p = -\frac{3\nabla\cdot\boldsymbol{u}^*}{2\Delta t}$$

- $\boldsymbol{u}^{n+1} = \boldsymbol{u}^* + \boldsymbol{u}'$
- $p^{n+1} = p^n + p'$
- $\nabla^2\boldsymbol{u}'$ term is neglected

Incompressible Navier-Stokes (NS) Equations, Cont.

- Fully pressure-velocity coupled direct (FPCD) approach

$$\begin{bmatrix} \frac{1}{\Delta t} - \frac{\Delta^{(u)}}{Re} & 0 & 0 & \partial_x \\ 0 & \frac{1}{\Delta t} - \frac{\Delta^{(v)}}{Re} & 0 & \partial_y \\ 0 & 0 & \frac{1}{\Delta t} - \frac{\Delta^{(w)}}{Re} & \partial_z \\ \partial_x & \partial_y & \partial_z & 0 \end{bmatrix} \begin{pmatrix} u \\ v \\ w \\ p \end{pmatrix} = \begin{pmatrix} f_u \\ f_v \\ f_w \\ 0 \end{pmatrix}$$

$+$ *boundary conditions for velocity*

$+$ *Dirichlet point for pressure*

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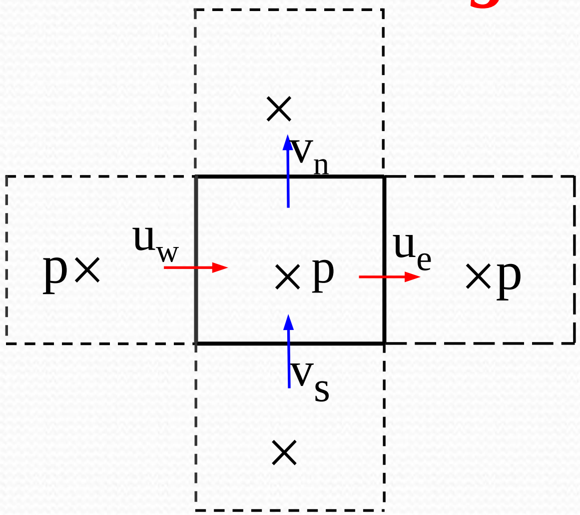
Solution of Incompressible Navier-Stokes (NS) Equations, Cont.1

S.P. Vanka (1985) – analytical solution for a ***single*** finite volume

- Vanka smoother based approach (1985) – analytical solution for a ***single*** finite volume



$$(u,v)^{\text{new}} = (u,v)^{\text{old}} + r_{(u,v)}(u,v)'$$

$$p^{\text{new}} = p^{\text{old}} + r_{\text{p}}\, p'$$

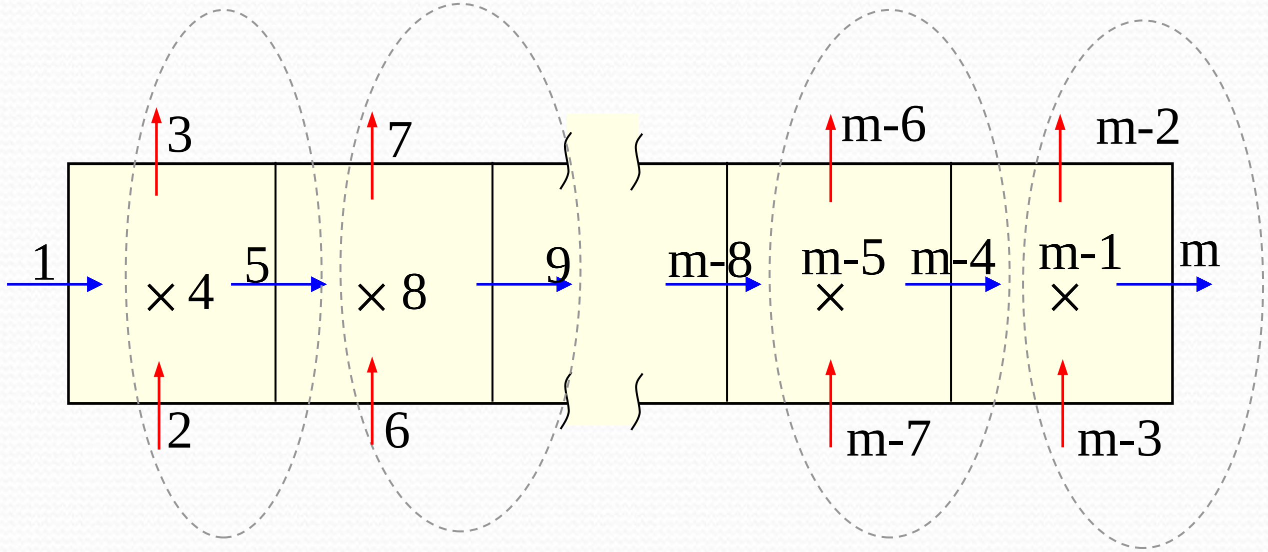
$$\begin{bmatrix} A_1 & 0 & 0 & 0 & A_2 \\ 0 & A_3 & 0 & 0 & A_4 \\ 0 & 0 & A_5 & 0 & A_6 \\ 0 & 0 & 0 & A_9 & A_{10} \\ A_7 & -A_7 & A_8 & -A_8 & 0 \end{bmatrix} \times \begin{bmatrix} u'_{\text{e}} \\ u'_{\text{w}} \\ v'_{\text{n}} \\ v'_{\text{s}} \\ p'_{\text{P}} \end{bmatrix} = \begin{bmatrix} R_{\text{ue}} \\ R_{\text{uw}} \\ R_{\text{vn}} \\ R_{\text{vs}} \\ R_{\text{cp}} \end{bmatrix}$$

for the Helmholtz operator and a constant time step A_i are constants and the 5×5 matrix can be inversed analytically

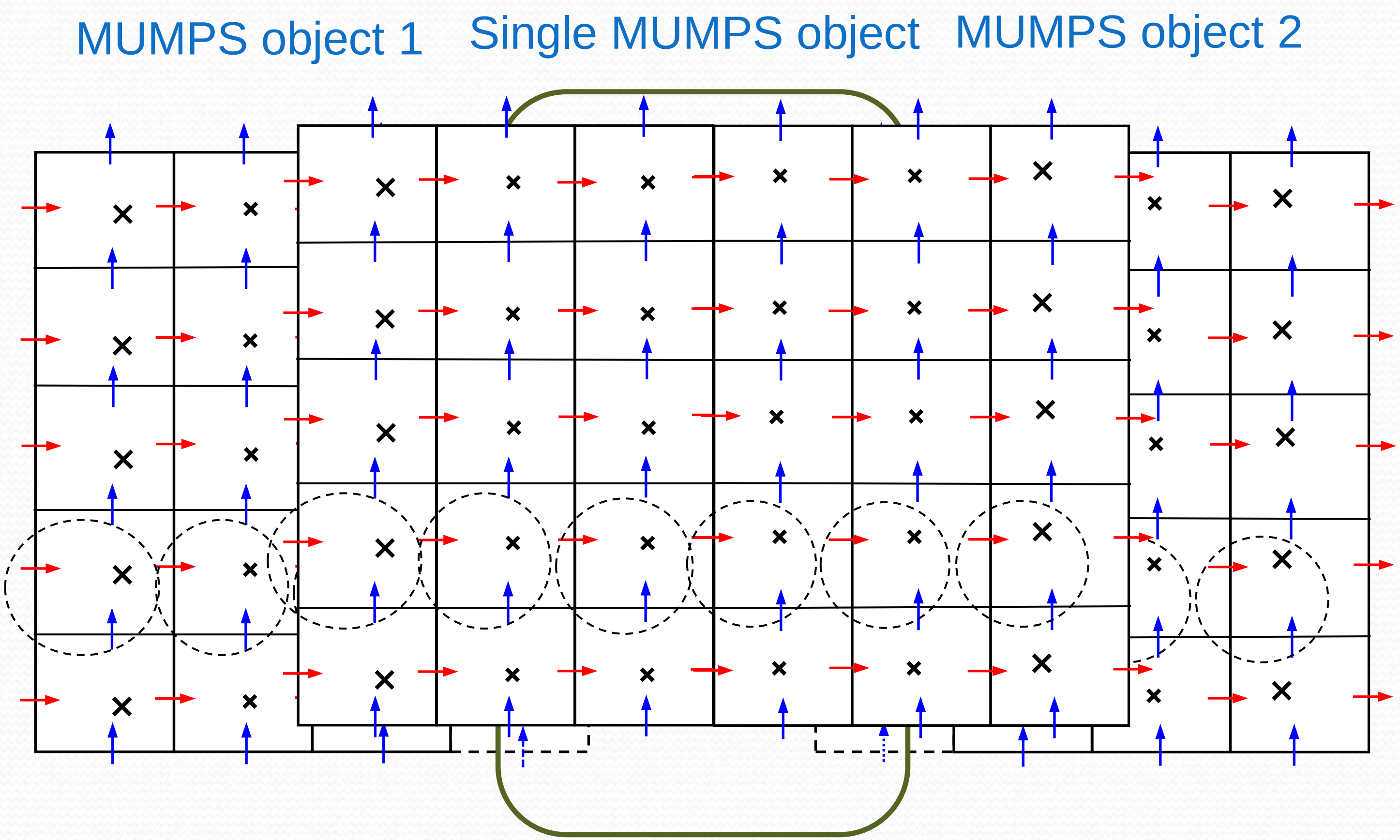
Solution of Incompressible Navier-Stokes (NS) Equations, Cont.2

- Accelerated Coupled Line Gauss-Seidel Smoother (ASA-CLGS)

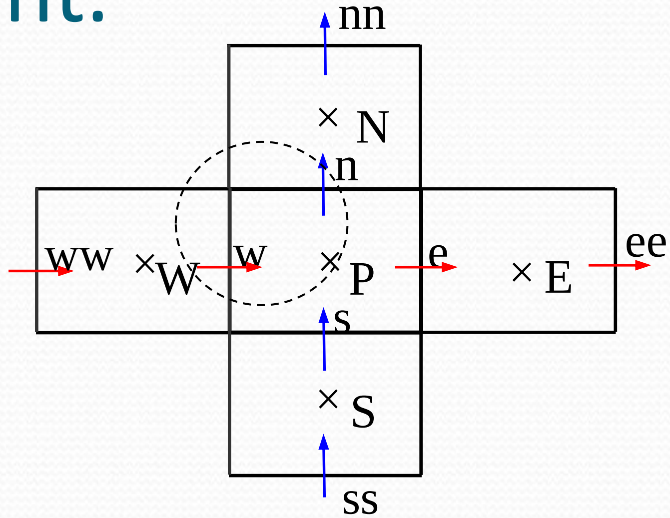
Zeng and Wesseling (1993) – CLGS   Feldman and Gelfgat (2009) – ASA-CLGS



Block Gauss-Seidel Method Accelerated by MUMPS



Block Gauss-Seidel Method Accelerated by MUMPS, Cont.



$$\left(\frac{3}{2}\Delta t+\frac{1}{Re}a_w\right)u'_w+a_P p'_P-a_W p'_W=\left[-\frac{\partial p}{\partial x}+\nabla^2 u+\nabla\cdot(u\boldsymbol{u})-\frac{3}{2}\Delta t\,u_w\right]^{(k)}+\left(2u_w^n-\frac{1}{2}u_w^{n-1}\right)\Delta t$$

$$\left(\frac{3}{2}\Delta t+\frac{1}{Re}a_s\right)u'_s+a_P p'_P-a_S p'_S=\left[-\frac{\partial p}{\partial y}+\nabla^2 v+\nabla\cdot(v\boldsymbol{u})-\frac{3}{2}\Delta t\,u_s\right]^{(k)}+\left(2u_s^n-\frac{1}{2}u_s^{n-1}\right)\Delta t$$

$$\frac{u'_e-u'_w}{\Delta x}+\frac{u'_n-u'_s}{\Delta y}=-\left(\frac{u_e-u_w}{\Delta x}+\frac{u_n-u_s}{\Delta y}\right)$$

$$\boldsymbol{u}^{k+1}=\boldsymbol{u}^k+\alpha\boldsymbol{u}';\ p^{k+1}=p^k+\alpha p'$$

Immersed Boundary (IB) Method

- Simulation of flows in the presence of complex geometries and moving boundaries.

- Straight forward computation of the forces/heat fluxes acting on the immersed boundary.

- The implementation of the method requires very limited modifications of the existing time stepping/linear stability codes.

Governing Equations

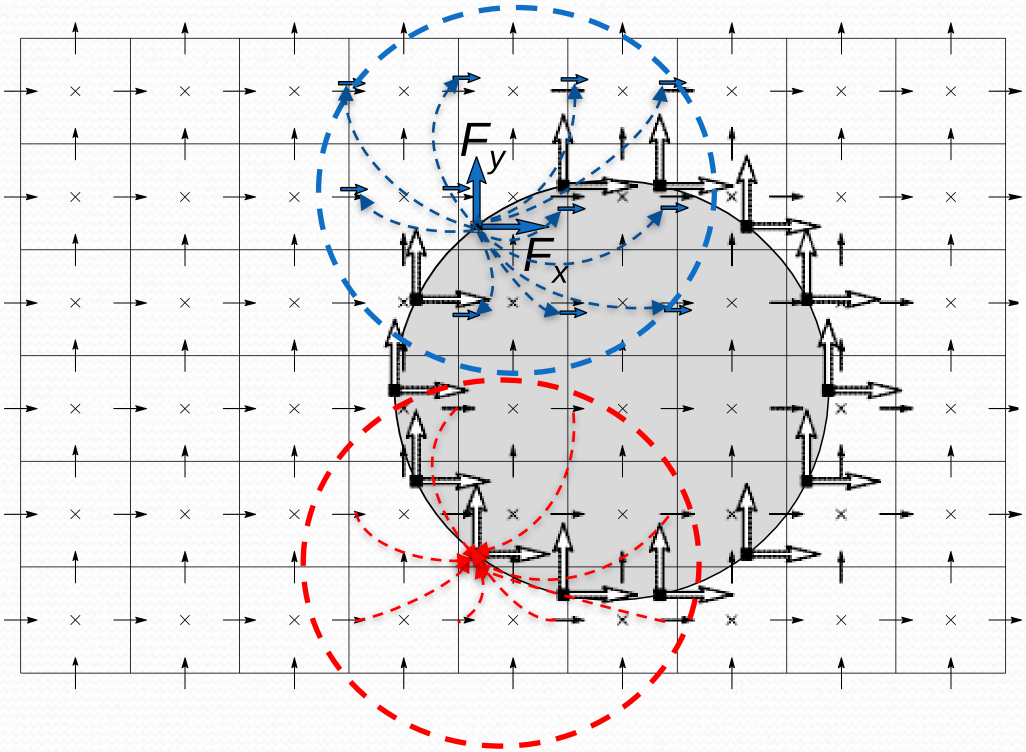
Continuity: $\nabla \cdot \boldsymbol{u} = 0$

Momentum: $$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{Re}\nabla^2 \boldsymbol{u} \quad + \boldsymbol{f}$$

Non-slip constraint: $\boldsymbol{u} = \boldsymbol{U}^{\Gamma}$

- No-slip boundary conditions
- Fully 3-D flow
- 2-nd order spatial and temporal discretization
- Threating the source force as distributed Lagrange multiplier to satisfy the kinematic non-slip constraints

Concept of IB Method



The same discrete Delta function
for regularization /interpolation

$$d(r) = \begin{cases} \frac{1}{6\Delta r}\left[5 - 3\frac{|r|}{\Delta r} - \sqrt{-3\left(1-\frac{|r|}{\Delta r}\right)^2 + 1}\right] & \text{for } 0.5\Delta r \leqslant |r| \leqslant 1.5\Delta r, \\ \frac{1}{3\Delta r}\left[1 + \sqrt{-3\left(\frac{r}{\Delta r}\right)^2 + 1}\right] & \text{for } |r| \leqslant 0.5\Delta r, \\ 0 & \text{otherwise,} \end{cases}$$

Roma et al. 1999

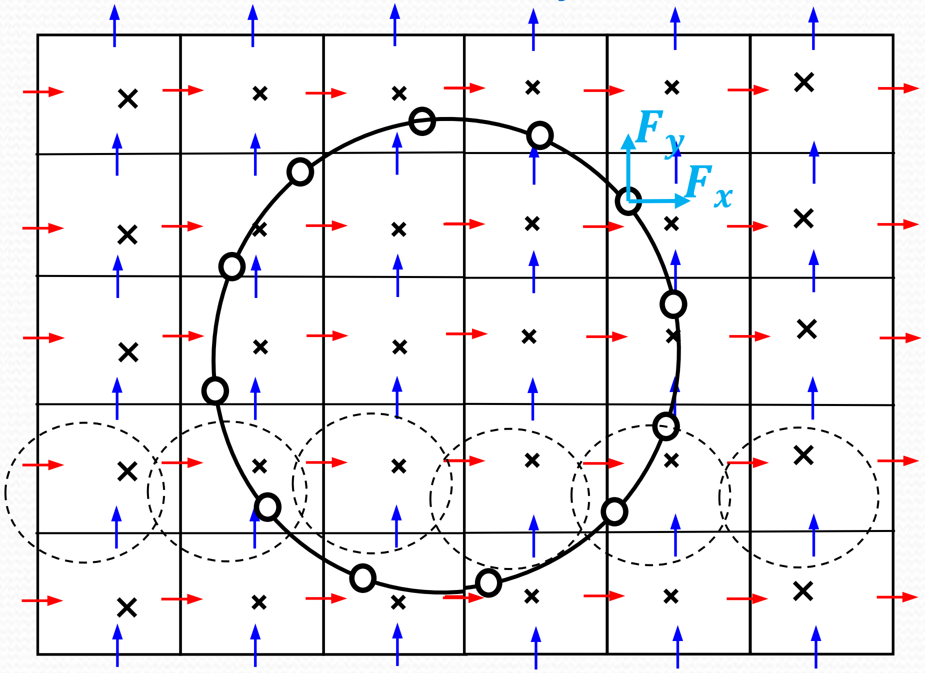
- Regularization operator to smear force.
- Interpolation operator to interpolate velocity

Fully Implicit Direct Forcing: Pros and Cons

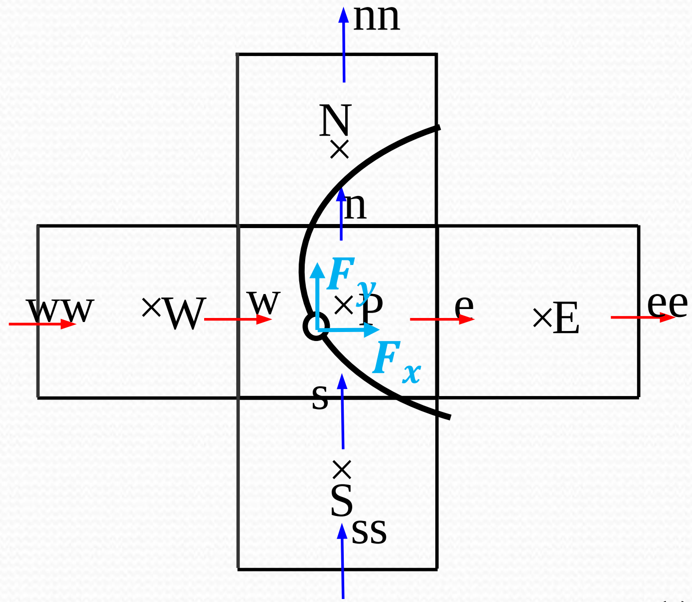
- The incompressibility and kinematic constraints are enforced implicitly up to the machine zero precision.
- Straight forward computation of the total forces and heat fluxes related to immersed body.
- Solution of the modified Poisson equation is expansive for realistic 3D problems.
- Straight-forward embedding into generic existing time stepping codes is often challenging

IBM—Block Gauss-Seidel Method Accelerated by MUMPS

MUMPS object



IBM—Block Gauss-Seidel method accelerated by MUMPS, Cont.



$$\left(\frac{3}{2}\Delta t+\frac{1}{Re}a_w\right)u'_w + a_P p'_P - a_W p'_W - \sum_i R[F'_{x_i}] = \left[-\frac{\partial p}{\partial x} + \nabla^2 u + \nabla\cdot(u\boldsymbol{u}) - \frac{3}{2}\Delta t\; u_w\right]^{(k)} + \left(2u_w^n - \frac{1}{2}u_w^{n-1}\right)\Delta t + \sum_i R[F_{x_i}^{(k)}]$$

$$\left(\frac{3}{2}\Delta t+\frac{1}{Re}a_s\right)u'_s + a_P p'_P - a_S p'_S - \sum_i R[F'_{y_i}] = \left[-\frac{\partial p}{\partial x} + \nabla^2 u + \nabla\cdot(u\boldsymbol{u}) - \frac{3}{2}\Delta t\; u_s\right]^{(k)} + \left(2u_s^n - \frac{1}{2}u_s^{n-1}\right)\Delta t + \sum_i R[F_{y_i}^{(k)}]$$

$$\frac{u'_e - u'_w}{\Delta x} + \frac{u'_n - u'_s}{\Delta y} = -\left(\frac{u_e - u_w}{\Delta x} + \frac{u_n - u_s}{\Delta y}\right)$$

$\sum_i I[u'] = U^\Gamma - \sum_i I[u^{(k)}]$ $\quad$ $\sum_i I[v'] = U^\Gamma - \sum_i I[v^{(k)}]$

$$\boldsymbol{u}^{k+1} = \boldsymbol{u}^k + \alpha\boldsymbol{u}';\; p^{k+1} = \alpha p^k + p'; \qquad \boldsymbol{F}^{k+1} = \boldsymbol{F}^k + \alpha\boldsymbol{F}'$$

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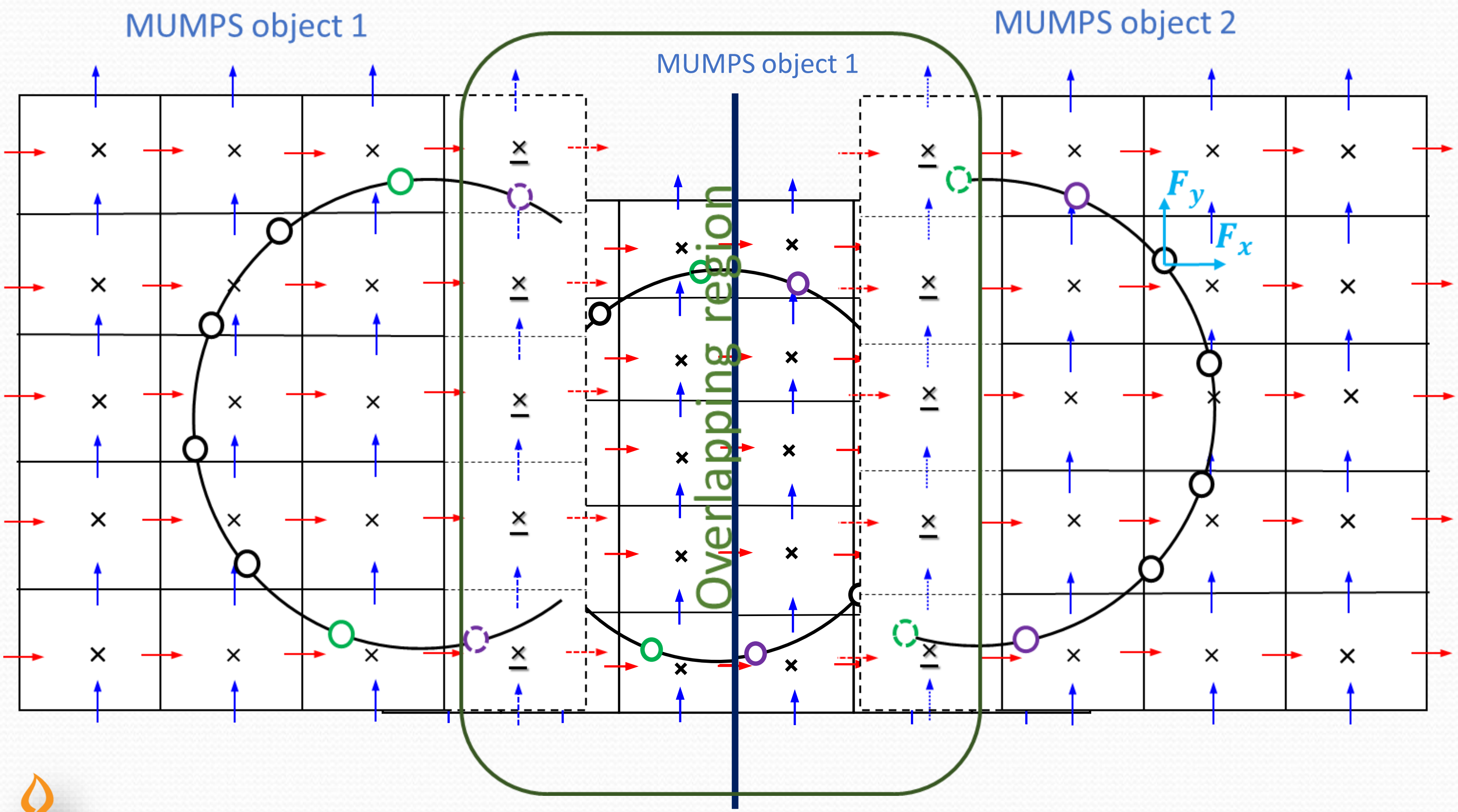
IBM—Block Gauss-Seidel method: Domain decomposition



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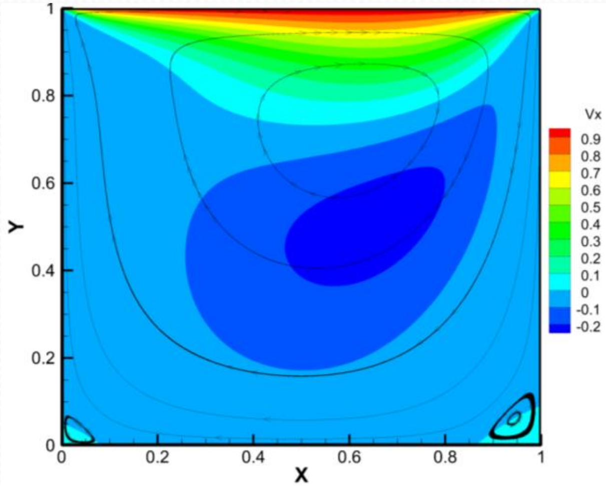
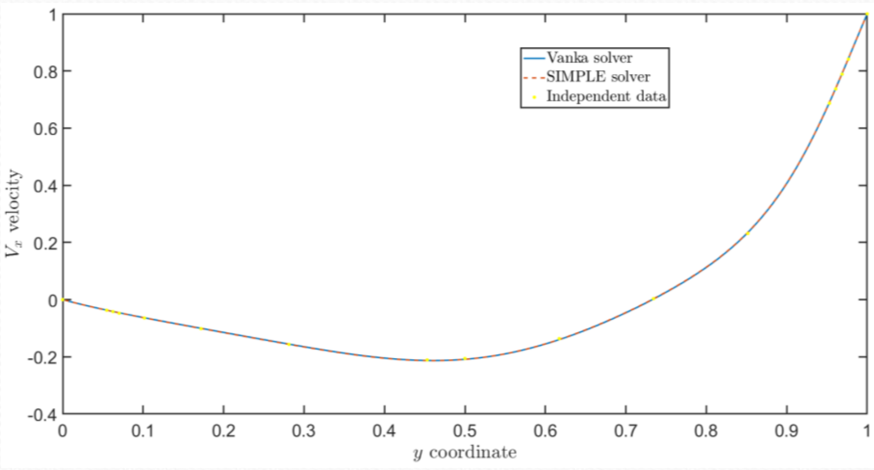
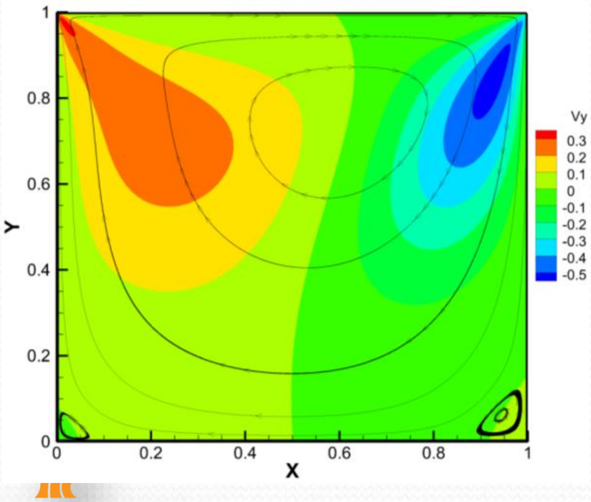
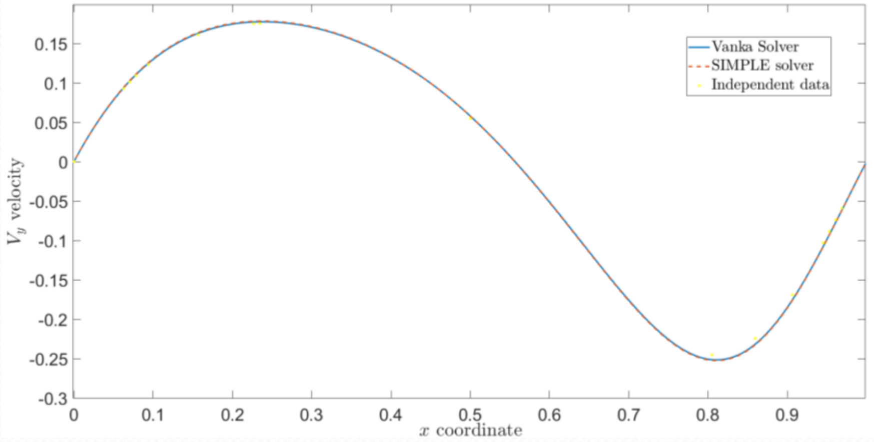
Benchmark Tests

- Lid driven cavity, $Re = 100$



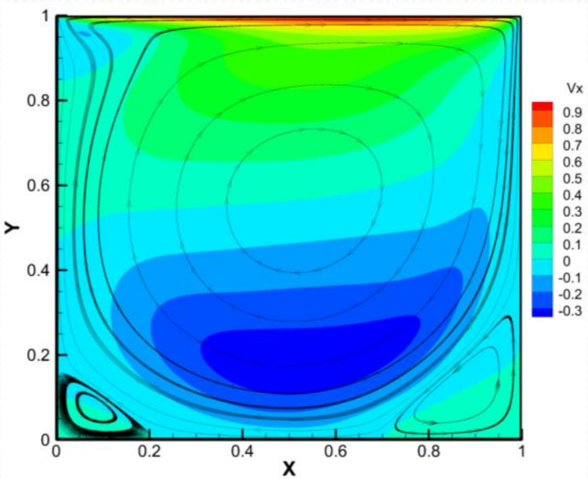


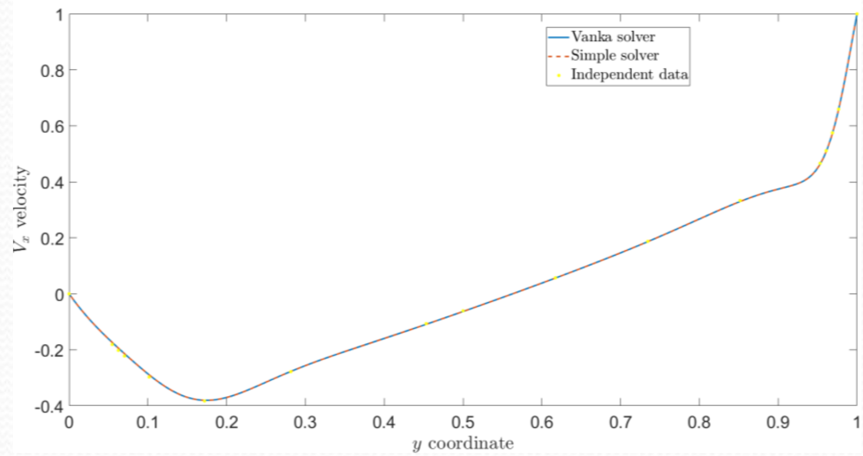


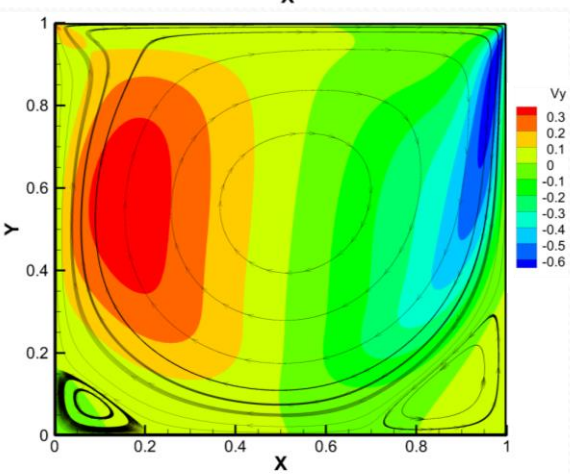


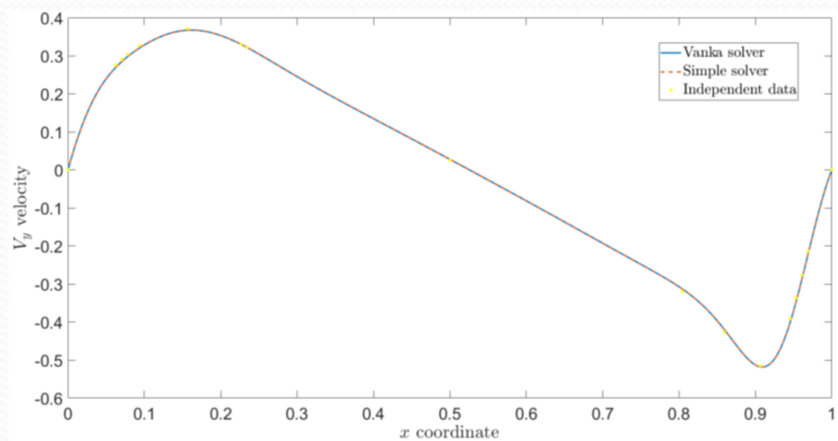
Benchmark Tests, Cont.

- Lid driven cavity, $Re = 1000$



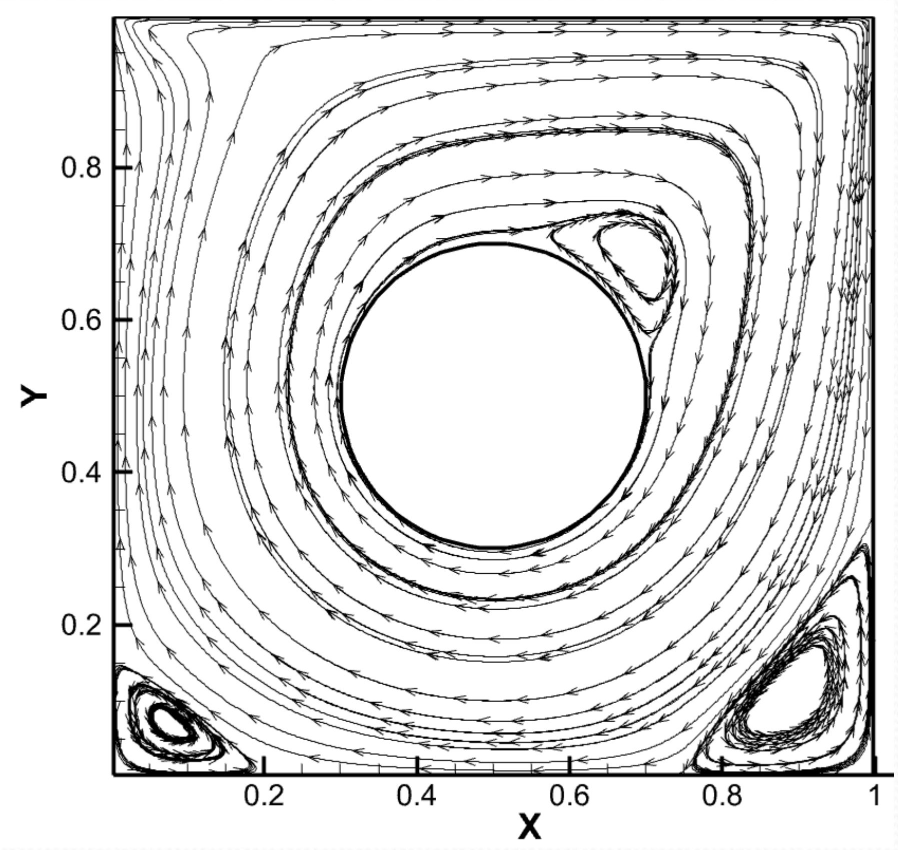


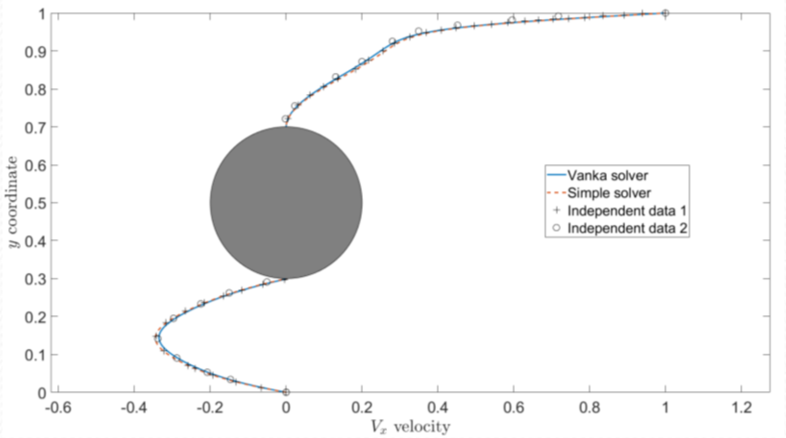


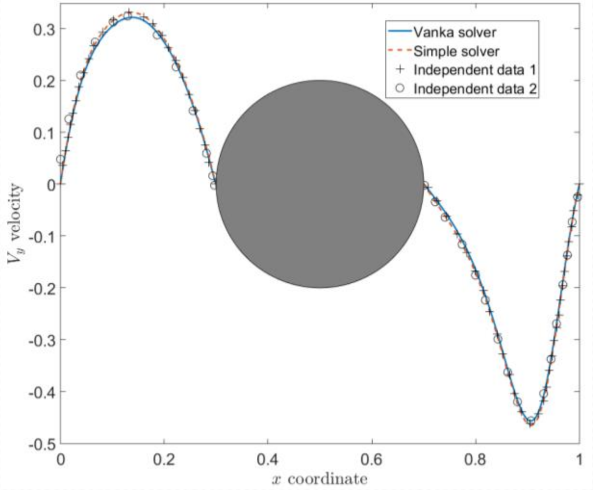


Benchmark Tests, Cont.1

- Lid driven cavity, $R = 0.2; Re = 1000$







What is next?

- Extension for simulating 3D flows (straight forward)
- Using MUMPS a smoother for the multigrid approach
- Extension for simulating moving boundary and two way coupled FSI problems
- Extension for simulating two- and multiphase flows